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Computation of Antifractals - Tricorns and Multicorns and Their Complex Nature

Narayan Partap[1]*, Sarika Jain[2] and Renu Chugh[3]

[1]*Northern India Engineering College*, *GGSIP University*, *New Delhi, 110053 India*

[2]*Amity Institute of Information Technology, Amity University, Uttar Pradesh*, *201313 India*

[3]*Department of Mathematics, M.D. University, Rohtak, Haryana*, *124001 India*

ABSTRACT

Since the last decade, the study of fractals and antifractals and their complex nature has been one of the significant areas of research. While many researchers are working on fractals, only a very few have focused on antifractals and their dynamic nature. This paper discovers new antifractals like tricorns for the complex antipolynomial $\bar{z}^n + c$ for $n \geq 2$ in the *GK*-orbit. The main focus of this paper is to develop a new collection of antifractals and to understand their attributes using the *GK*-orbit.

Keywords: Antifractals, antipolynomial, *GK*-orbit, multicorns, tricorns

INTRODUCTION

Anti-Julia, tricorns and multicorns are some of the striking examples of antifractals that can be generated through the dynamics of antiholomorphic complex polynomials $\bar{z}^n + c$ of the complex polynomial $z^n + c$ for $n \geq 2$. They have been named antifractals as tricorns and multicorns are generated through antiholomorphic complex polynomials $\bar{z}^n + c$ for $n \geq 2$. In mathematics, we define tricorn in a similar way as the Mandelbrot set, but we use the $z \to \bar{z}^2 + c$ map instead of the $z \to z^2 + c$ map, the map used for the Mandelbrot set; therefore, sometimes the tricorn is also called the Mandelbar set. The tricorn was initiated by Crowe, Hasson, Rippon and Strain-Clark (1989). They worked on the antiholomorphic complex polynomial $\bar{z}^n + c$ and studied the bifurcation diagram for iterates of the above map. These tricorns and multicorns possess various attributes. The three-cornered nature, the main characteristic of a tricorn, repeats with variations at different scales, reflecting the same sort of self-similarity as the Mandelbrot set.

Apart from complications and their beauty, the same may be connected or disconnected. The set of all parameters c for

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E-mail addresses:
narayan.maths81@gmail.com (Narayan Partap)
ashusarika@gmail.com (Sarika Jain)
chugh.r1@gmail.com (Renu Chugh)
*Corresponding Author

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which the Julia set of $\overline{z}^2 + c$ is connected, is called a multicorn. These multicorns are simply higher-order tricorns and they all are used in the commercial field.

Milnor (1999) coined the term 'tricorn' for the connectedness locus for the antiholomorphic polynomial $\overline{z}^2 + c$, which plays an intermediate role between quadratic and cubic polynomials. The tricorn has many similarities with the Mandelbrot set due to a compact subset of *C*.

Crowe et al. (1989) used the complex polynomial as a formal analogy with Mandelbrot sets and they called it the Mandelbar set, which has many similarities to the Mandelbrot set. They also brought their bifurcation features along arcs rather than at points. Milnor (1999) found multicorns in a real slice of the cubic connectedness locus. Winters (1990) discovered the boundary of the tricorn, showing that it contains a smooth arc. Nakane and Schleicher (2003) revealed the various properties of tricorns and multicorns along with their complexity and their beauty and stated that multicorns are generalised tricorns or tricorns of higher order. They also explored the antifractal of a complex polynomial of the form $M_c(z) = \bar{z}^n + c$ for $n \geq 2$, showing that it is either connected or disconnected. The set of parameters c for which the Julia set of $M_c(z)$ is connected is called the multicorn. Lau and Schleicher (1996) analysed the symmetries of tricorns and multicorns.

Rani and Kumar (2009) studied the dynamics of the complex polynomial $M_c(z) = \bar{z}^n + c$ for $n \geq 2$ and Rani (2010) introduced superior tricorns and superior multicorns using Mann's iterative technique. In the same year, Chauhan, Rana and Negi (2010) introduced relative superior tricorns and relative superior multicorns using the Ishikawa iterative techniques. Kumar, Chugh and Rani (2012) generated a new class of tricorns and multicorns in the Noor orbit. Recently, Kang, Rafik and Latif (2015) presented tricorns and multicorns using the *S*-iteration scheme.

In this paper, we explore a new class of tricorns and multicorns in the *GK*-orbit and analyse their mathematical characteristics. These generated anti-fractals such as anti-Julia, tricorns and multicorns have their own attraction and are different from previously generated antifractals. Tricorn prints are being used for commercial objects like tricorn coffee cups, jugs and tricorn apparels like tricorn dresses and tricorn shirts, among others.

METHOD

Before generating a new class of tricorns and multicorns for the complex polynomial $M_c(z) = \bar{z}^n + c$, we need to define the basic terminology of the research.

Julia Set

The filled-in Julia set of the function $F(z) = z^n + c$ is defined as:

$$K(F) = \{z \in \mathbb{C} : F^k(z) \text{ does not tend to } \infty \},$$

where, $\mathbb{C}$ is the complex space, $F^k(z)$ is the k^{th} iterate of function *F* and $K(F)$ denotes the filled Julia set. The Julia set of the function *F* is defined to the boundary of $K(F)$, that is, $J(F) = \partial K(F)$, where $J(F)$ denotes the Julia set. The set of points whose orbits are bounded under $F_c(z) = z^2 + c$ is called the Julia set (Peitgen, Jurgens, & Saupe, 1994).

The Julia and Mandelbrot sets, which have great importance in fractal geometry, can be generated through fixed points using different complex maps (Wang & Shi, 2006; Nazeer, Kang, Tanveer, & Shahid, 2015).

Multicorn

The multicorn for the complex function $M_c(z) = \bar{z}^n + c$ is defined as the collection of all $c \in \mathbb{C}$, for which orbit of the point 0 is bounded, that is:

$$M_c = \{c \in \mathbb{C} : M_c^n(0) \text{ does not tend to } \infty \},$$

where, $\mathbb{C}$ is the complex space and $M_c^n(z)$ is the n[th] iterate of function $M_c(z)$. An equivalent formulation is the connectedness of loci for higher degree antiholomorphic polynomials $M_c(z) = \bar{z}^n + c$ that are called multicorns (Devaney, 1992).

It has been observed that for $n = 2$, multicorns reduce to tricorns. Moreover, the tricorns naturally live in the real slice $d = \bar{c}$ in the two-dimensional parameter space of map $z \to (z^2 + d)^2 + c$. They have $(n + 1)$-fold rotational symmetries. After dividing these symmetries, the resulting multicorns are called unicorns (Nakane & Schleicher, 2003).

GK-Orbit

Consider f as a self-map from a metric space X into itself such that $f : X \to X$. Then the generalised Kransnoselskii iteration scheme (Schaefer, 1957) is defined as:

$$x_{n+1} = (r_n)x_n + (1 - r_n)f(x_n), \qquad n = 0,1,2,3,\ldots$$

where, $0 \leq r_n < 1$ and $< r_n >$ are a sequence of positive numbers.

For $x_0 \in X$ (real or complex numbers), construct a sequence $< x_n >$ in X in the following manner:

$$x_1 = r_1 x_0 + (1 - r_1)f(x_0),$$

$$x_2 = r_2 x_1 + (1 - r_2)f(x_1),$$

$$x_n = r_n x_{n-1} + (1 - r_n)f(x_{n-1}),$$

where, $0 \leq r_n < 1$ and $\{r_n\}$ are a sequence converging to non-zero numbers. Then the sequence $x_1, x_2\ x_3 \ldots\ x_n$ constructed above is called the *GK*-orbit. We may also denote as *GKO* (f, x_0, r_n).

To visualize antifractals in the *GK*-orbit for $z \to \bar{z}^n + c$, we need an escape criterion with respect to the *GK*-orbit. The escape criterion for $z \to \bar{z}^n + c$ in *GK*-orbit is $max\,\{|c|, [2/(1 - r)]^{1/(n-1)}\}$ for $0 \leq r_n < 1$.

Escape Criterions for Antifractals

To prove the general escape criterion, we first prove the following result for the cubic complex polynomial.

Escape criterion for cubic polynomial. Suppose for a function $M_{a,b} = \bar{z}^3 + a\bar{z} + b$, $|\bar{z}| \geq |b| > \{|a| + 2/(1-r)\}^{1/2}$, where, $0 \leq r < 1$ and a, b are complex numbers.

Define

$$z_1 = (1-r)M_{a,b}(z) + r\bar{z}$$

$$z_2 = (1-r)M_{a,b}(z_1) + r\bar{z_1}$$

$$\vdots$$

$$z_n = (1-r)M_{a,b}(z_{n-1}) + r\bar{z}_{n-1}, \quad n = 2, 3, \ldots$$

Then $|z_n| \to \infty \; as \; n \to \infty$

This is proved by definition:

$$|z_1| = \left|(1-r)M_{a,b}(z) + r\bar{z}\right|$$

$$= |(1-r)\,(\bar{z}^3 + a\bar{z} + b)\; + r\bar{z}|$$

$$= |(1-r)\bar{z}^3 + a(1-r)\bar{z} + r\bar{z} + (1-r)b\,|$$

$$\geq |(1-r)\bar{z}^3 + a(1-r)\bar{z} + r\bar{z}| - |(1-r)b\,|$$

$$\geq |\bar{z}||(1-r)\bar{z}^2 + a(1-r) + r| - |(1-r)\bar{z}\,|$$

$$\geq |\bar{z}|(|(1-r)\bar{z}^2 + a(1-r)| - r) - |(1-r)\bar{z}|$$

$$= |\bar{z}|(|(1-r)\bar{z}^2 + a(1-r)| - r - (1-r))$$

$$= |\bar{z}|\{(1-r)|\bar{z}^2 + a| - 1\}$$

$$= |\bar{z}|(1-r)\{|\bar{z}^2 + a| - 1/(1-r)\}$$

$$\geq (1-r)|\bar{z}|[|\bar{z}|^2 - \; |a| - 1/(1-r)]$$

As $|\bar{z}| > \{|a| + 2/(1-r)\}^{1/2}$.

Therefore,

$$|\bar{z}|^2 - \; |a| - 1/(1-r)\} > 1/(1-r)$$

$$(1-r)[|\bar{z}|^2 - \; |a| - 1/(1-r)] > 1$$

Hence, we can find a $\gamma > 1$ such that

$$|z_1| > \lambda|\bar{z}|.$$

Repeating this argument *n* time, we find $|z_n| > \lambda^n|\bar{z}|$.

Therefore, the *GK*-orbit of $\bar{z}$ under the cubic polynomial $M_{a,b}(z)$ tends to infinity, and max $\left\{|b|, [|a| + 2/(1-r)]^{1/2}\right\}$ is the required escape criterion. This completes the proof.

General escape criterion. Assume the function of the form $M_c(z) = \bar{z}^n + c$, n =1, 2, 3… where $0 \le r < 1$ and *c* is in the complex plane, Define

$$z_1 = (1-r)M_c(z) + r\bar{z}$$

$$z_2 = (1-r)M_c(z_1) + r\bar{z}_1$$

$$\vdots$$

$$z_n = (1-r)M_c(z_{n-1}) + r\bar{z}_{n-1},\ n = 2, 3, \ldots$$

Then, the general escape criterion is max $\{|c|, [2/(1-r)]^{1/(n-1)}\}$.

We shall prove the result by induction.

For $n = 1$, $M_c(z) = \bar{z} + c$ so, the escape criterion is $|c|$, which is obvious i.e. $|z| > \max\{|c|, 0\}$.

For $n = 2$, $M_c(z) = \bar{z}^2 + c$

so, the escape criterion is *max* $\{|c|, 2/(1-r)\}$, which can be easily proved.

For $n = 3$, $M_c(z) = \bar{z}^3 + c$

This result follows from above theorem with $a = 0$ and $b = c$ i.e. the escape criterion for $\bar{z}^3 + c$ is $\max\{|c|, [2/(1-r)]^{1/2}\}$; thus, the theorem is true for $n = 1, 2,$ and 3.

Now, suppose the theorem is true for any *n*, we shall prove it for *n* + 1.

Suppose $M_c(z) = \bar{z}^{n+1} + c$ and $|\bar{z}| \ge |c| > [2/(1-r)]^{1/n}$

Then, $|z_1| = |(1-r)M_c(z) + r\bar{z}|$

$$= |(1-r)(\bar{z}^{n+1} + c) + r\bar{z}|$$

$= |\bar{z}^{n+1}(1-r)\ + r\bar{z} + (1-r)c|$

$\geq |\bar{z}^{n+1}(1-r)\ + r\bar{z}| - |(1-r)c|$

$\geq |\bar{z}|[|\bar{z}^{n}(1-r)\ + r|] - |(1-r)\bar{z}|$

$\geq |\bar{z}|[|(1-r)\bar{z}^{n}| - r] - |(1-r)\bar{z}|$

$= |\bar{z}|[|(1-r)\bar{z}^{n}| - 1]$

As $|\bar{z}| > [2/(1-r)]^{1/n}$, we can find a $\lambda > 0$ such that $(1-r)|\bar{z}|^{n} - 1 > 1 + \lambda$

Therefore,

$|z_1| > (1+\lambda)|\bar{z}|$.

Repeating this argument, we find

$|z_n| > (1+\lambda)^{n}|\bar{z}|$

Thus, the *GK*-orbit of $\bar{z}$ under the general iteration function $\bar{z}^{n+1} + c$ tends to infinity. Therefore, max $\{|c|,\ [2/(1-r)]^{1/n}\}$ is the required escape criterion. This completes the proof.

Corollary. Suppose that $|c| > [2/(1-r)]^{1/n-1}, 0 \leq r < 1$. Then the *GK*-orbit *GKO* $(G_c, 0, r)$ escapes to infinity i.e. orbit of 0 escapes to infinity under M_c(z).

General algorithm. Assume that for some $k \geq 0$, we have $|z_k| > max\ \{|c|, [2/(1-r)]^{1/(k-1)}\}$. Then $|z_{k+1}| > (1+\gamma)|z_k|$ and $|z_n| \to \infty$ as $n \to \propto$

The above result provides a general algorithm for computing the filled *GK* Julia set for the function $M_c(z) = \bar{z}^n + c$, n=1, 2, 3…

RESULTS AND DISCUSSION

Tricorns and Multicorns

Antifractals, tricorns and multicorns in *GK*-orbit, were generated for the complex polynomial $M_c(z) = \bar{z}^n + c$ using the general escape criterion with the software Mathematica 9.0 (see Figures 1-12).

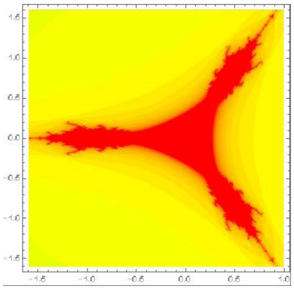

Figure 1.
Tricorn for $r = 0.001$, $n = 2$

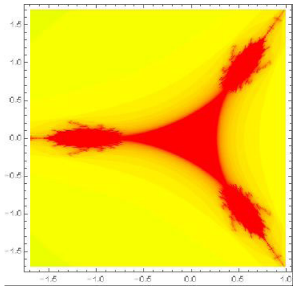

Figure 2.
Tricorn for $r = 0.031$, $n = 2$

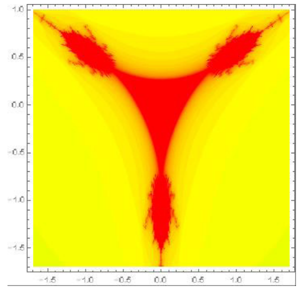

Figure 3.
Tricorn for $r = 0.051$, $n = 2$

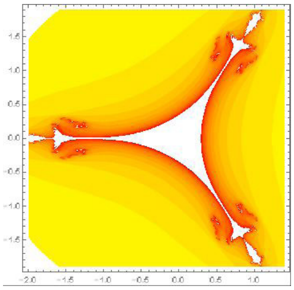

Figure 4.
Tricorn for $r = 0.191$, $n = 2$

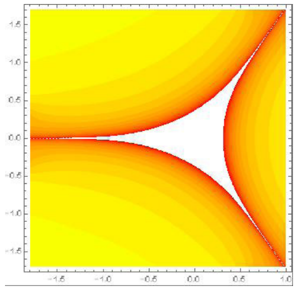

Figure 5.
Tricorn for $r = 0.391$, $n = 2$

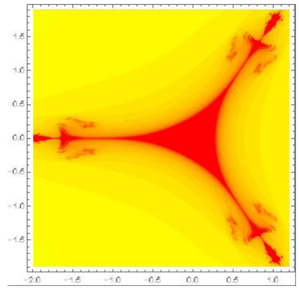

Figure 6.
Tricorn for $r = 0.195$, $n = 2$

The following characteristics were noticed after the generation of tricorns and multicorns in the *GK*-orbit:

- It was observed that branches in the tricorns and multicorns were one more than the degree i.e. the number of branches was n+1, if n is the power of $\bar{z}$. Also, some branches had n sub-branches (Figures 1-3).
- The multicorns exhibited (n+1)-fold rotational symmetries.

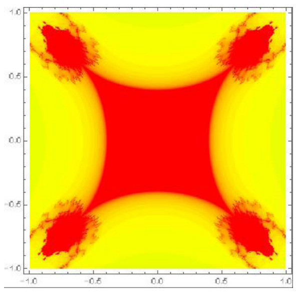

Figure 7.
Multicorn for $r = 0.091$, $n = 3$

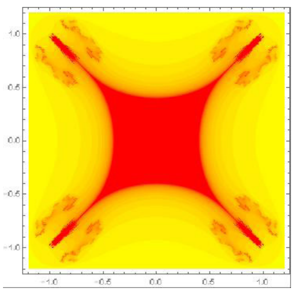

Figure 8.
Multicorn for $r = 0.191$, $n = 3$

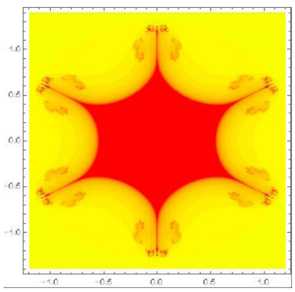

Figure 9.
Multicorn for $r = 0.291$, $n = 5$

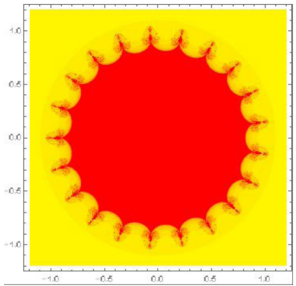

Figure 10.
Multicorn for $r = 0.151$, $n = 20$

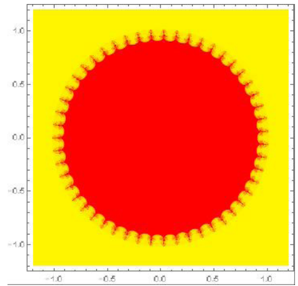

Figure 11.
Multicorn for $r = 0.151$, $n = 50$

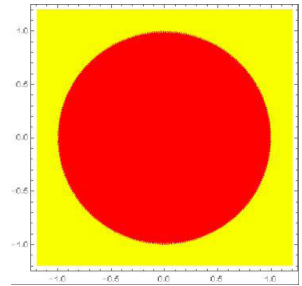

Figure 12.
Multicorn for $r = 0.01$, $n = 374$

- It was also observed that if we increased the value of r, the tricorns in Figures 4-6 and the multicorns in Figures 7-9 became thinner.
- Figures 3-6 and Figures 7-8 showed that for a fixed value of n, there existed many antifractals.
- If we increased the degree of the complex polynomial, the generated multicorns became a circular saw (Figures 10-12). Kumar at al. (2012) came to a similar conclusion while generating tricorns and multicorns using different iterates. The name circular saw was first considered by Rani and Kumar (2009) for Mandelbrot sets.

Anti-Julia Sets in GK-Orbit

Anti-Julia sets in *GK*-orbit were generated with the help of a general escape criterion for the complex polynomial $M_c(z) = \bar{z}^n + c$ using the software Mathematica 9.0 (see Figures 13-21).

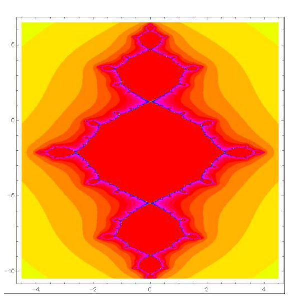

Figure 13.
Anti-Julia for $r = 0.809$,
$c = -35.505-0.001I$, $n = 2$

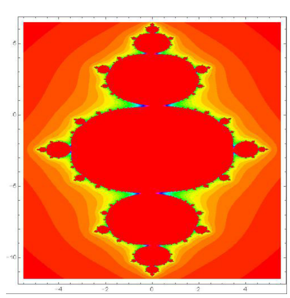

Figure 14.
Anti-Julia for $r = 0.809$,
$c = -36.505-0.071I$, $n = 2$

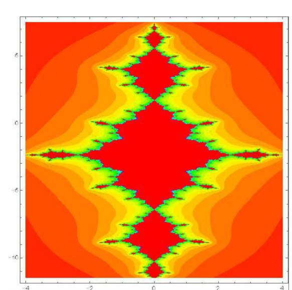

Figure 15.
Anti-Julia for $r = 0.815$,
$c = -47.505-0.001I$, $n = 2$

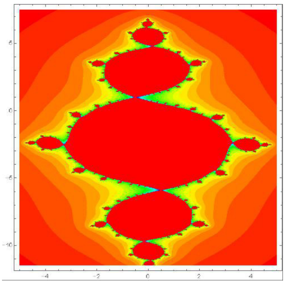

Figure 16.
Anti-Julia for $r = 0.834$,
c = -40.505+0.5I, $n = 2$

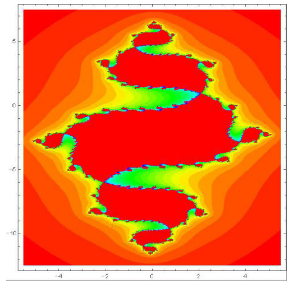

Figure 17.
Anti-Julia for $r = 0.834$,
c = -37.505+1.95I, $n = 2$

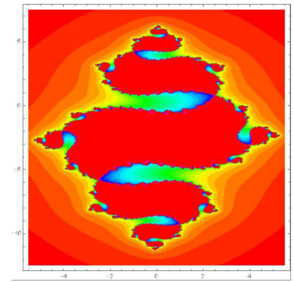

Figure 18.
Anti-Julia for $r = 0.834$,
c = -33.505+1.55I, $n = 2$

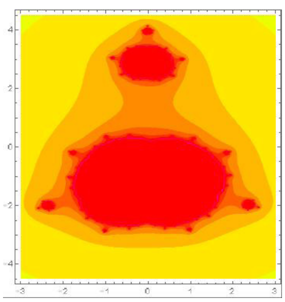

Figure 19.
Anti-Julia for $r = 0.860$,
c = -0.60-17.20I, $n = 3$

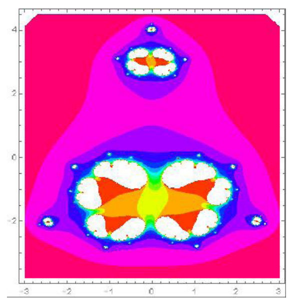

Figure 20.
Anti-Julia for $r = 0.860$,
c = -0.60-19.20I,
$n = 3$

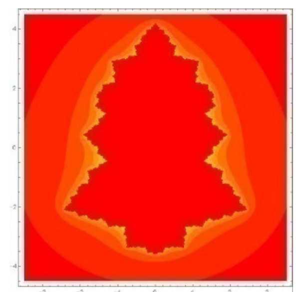

Figure 21.
(Christmas tree)
Anti-Julia for $r = 0.879$,
c = 0.002-11.50I, $n = 3$

The following observations were made:

- Figures 13-15 show that if we fixed the parameter r and decreased the value of m in $c = m + n\,i$, the anti-Julia sets became more gracious. Similarly, Figures 16-18 show that if we fixed the parameter r and increased the value of m in $c = m + n\,i$, the connected anti-Julia sets in these figures looked like a helix.
- The anti-Julia sets in Figures 13-18 and in Figure 21 were connected, whereas the anti-Julia sets in Figures 19-20 were disconnected.
- The anti-Julia sets in Figures 17-18 show the reflection symmetry for the real axis point, whereas the anti-Julia sets in Figures 13-15 show the reflection symmetry for both the axes.
- The anti-Julia sets in Figures 19-20 looked like the meditation posture.
- The anti-Julia set in Figure 21 looked like a Christmas tree and showed the reflection symmetry for the imaginary axis.

CONCLUSION

In this paper, a new iterative procedure was employed in the antiholomorphic complex polynomials $\bar{z}^n + c$ of the complex polynomial $z^n + c$ for $n \geq 2$ and important as well as attractive antifractals like tricorns and multicorns were generated. It was fascinating to see that some of the generated antifractals were connected, while some were disconnected, and some anti-Julia sets were shaped like a helix and one anti-Julia set looked like a Christmas tree.

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